

# Finslerian Anisotropic Relativistic Metric Function Obtainable under Breakdown of Rotational Symmetry

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## Abstract

We undertake to show how the relativistic Finslerian Metric Function (FMF) should arise under uni-directional violation of spatial isotropy, keeping the condition that the indicatrix (mass-shell) is a space of constant negative curvature. By evaluating respective Finslerian tetrads, and treating them consistently as the bases of inertial reference frame (RF), the generalized Finslerian kinematic transformations follow in a convenient explicit form. The concomitant Finslerian relativistic relations generalize their Lorentzian prototypes through the presence of one characteristic parameter  $g$ , so that the constraints on the parameter may be found in future high-precision post-Lorentzian experiments. As the associated Finslerian Hamiltonian function is also obtainable in a clear explicit form, convenient prospects for the Finslerian extension of particle dynamics are also opened. Additionally, the Finslerian extension of the general-relativistic Schwarzschild metric can unambiguously be proposed.

## 1. Introduction

Minkowski [1,2] is well-known to have noted that the Special Relativity (SR) implies introducing the pseudo-Euclidean metric to geometrize the space-time. After that, the concepts of the SR and of the space-time pseudo-Euclidean geometry became living as two faces of the same cone. This event of history impels one to draw various farther-reaching conclusions and, first of all, that a researcher cannot be so self-determinant as to believe that one is able to create new self-consistent and universal-meaning theory of SR without inscribing it into the realms of some new particular geometry introduced over space-time. For a fundamental geometry presupposition is required. In this connection, the Finsler geometry (FG) seems to serve as being just the nearest appropriate metric extension of the pseudo-Euclidean geometry.

The spatial symmetry lies at the root of the ordinary SR, matching in fact the relevant symmetry of the pseudo-Euclidean metric. However, since beyond square-root metric one may apply ingenious methods of FG [3,4]. the Finslerian approach may be hoped to propose the post-Lorentzian kinematic transformations compatible with the possibility of breakdown of the spatial symmetry. The FMF  $F$  obtained and considered below in Section 2 refers to situation in which there is an uni-directional breakdown. The associated Finslerian Hamiltonian function  $H$  can be derived from  $F$  in an explicit way. The calculation of the associated curvature tensor leads to the remarkable conclusion that the indicatrix is a space of constant negative curvature.

In Section 3 we reveal various useful relations which get simplify the calculations involved. Relevant co-treatment is presented in Section 4. The meaning of the reference frame (FR) is not the same as the meaning of the reference system (RS). Elucidating the distinctions among them needs the use of the metric tensor tetrads. They will be calculated in Section 5, which enables us to derive in Section 6 the explicit form of the Finsler-generalized kinematic transformations. In the last Section 7, comments and conclusions on the general-relativistic theoretical aspects of the problem are presented, noting a handy possibility to continue Schwarzschild line element in due Finslerian domain.

## 2. Initial definitions and main observations

Suppose we are given an  $N$ -dimensional vector space  $V_N$ . Denote by  $R$  the vectors constituting the space, so that  $R \in V_N$ . Any given vector  $R$  assigns a particular direction in  $V_N$ . Let us fix a member  $R_{(1)}$  and introduce the straightline  $e_1$  oriented along the vector  $R_{(1)}$ , and use this  $e_1$  to serve as a  $R^1$ -coordinate axis in  $V_N$ . In this way we get the topological product

$$V_N = e_1 \times V_{N-1} \quad (2.1)$$

together with the separation

$$R = \{R^1, \mathbf{R}\}, \quad R^1 \in e_1 \quad \text{and} \quad \mathbf{R} \in V_{N-1}. \quad (2.2)$$

Also, we introduce a pseudo-Euclidean metric

$$q = q(\mathbf{R}) \quad (2.3)$$

over the  $(N - 1)$ -dimensional vector space  $V_{N-1}$ .

With respect to an admissible coordinate basis  $\{e_a\}$  in  $V_{N-1}$ , we obtain the coordinate representations

$$R = \{R^p\} = \{R^0, R^a\} \quad (2.4)$$

and

$$q(\mathbf{R}) = \sqrt{|r_{ab}R^aR^b|}, \quad (2.5)$$

where  $r_{ab}$  are the components of a symmetric  $(+ - \dots -)$ -indefinite tensor defined over  $V_{N-1}$ . The indices  $(a, b, \dots)$  and  $(p, q, \dots)$  will be specified over the ranges  $(0, 2, \dots, N-1)$  and  $(0, 1, \dots, N-1)$ , respectively; vector indices are up, co-vector indices are down; repeated up-down indices are automatically summed; the notation  $\delta_b^a$  will stand for the Kronecker symbol;  $N = 4$  in the physical space-time context proper. Sometimes we shall mention the associated post-Euclidean metric tensor

$$R_{pq} = \{R_{11} = -1, R_{1a} = 0, R_{ab} = r_{ab}\} \quad (2.6)$$

meaningful over the whole base space  $V_N$ .

The approach can be converted into the dual co-framework

$$\hat{V}_N = e^1 \times \hat{V}_{N-1} \quad (2.7)$$

to use the separation

$$\hat{R} = \{R_1, \hat{\mathbf{R}}\}, \quad R_1 \in e^1 \quad \text{and} \quad \hat{\mathbf{R}} \in \hat{V}_{N-1}. \quad (2.8)$$

With respect to the co-basis  $\{e_a\}$  in  $\hat{V}_{N-1}$ , we obtain the coordinate representation

$$\hat{R} = \{R_p\} = \{R_1, R_a\} \quad (2.9)$$

together with

$$\hat{q}(\hat{\mathbf{R}}) = \sqrt{|r^{ab}R_aR_b|}, \quad (2.10)$$

where  $r^{ab}r_{ba} = \delta_c^a$ . Also, the tensor

$$R^{pq} = \{R^{11} = -1, R^{1a} = 0, R^{ab} = r^{ab}\} \quad (2.11)$$

is meaningful over the whole vector space  $\hat{V}_N$ .

To ease calculations we introduce the convenient notation:

$$G = g/h, \quad (2.12)$$

$$h \stackrel{\text{def}}{=} \sqrt{1 + \frac{1}{4}g^2}, \quad (2.13)$$

$$g_+ = -\frac{1}{2}g + h, \quad g_- = -\frac{1}{2}g - h, \quad (2.14)$$

$$G_+ = g_+/h \equiv -\frac{1}{2}G + 1, \quad G_- = g_-/h \equiv -\frac{1}{2}G - 1, \quad (2.15)$$

$$g^+ = 1/g_+ = -g_-, \quad g^- = 1/g_- = -g_+, \quad (2.16)$$

$$g^+ = \frac{1}{2}g + h, \quad g^- = \frac{1}{2}g - h, \quad (2.17)$$

$$G^+ = g^+/h \equiv \frac{1}{2}G + 1, \quad G^- = g^-/h \equiv \frac{1}{2}G - 1, \quad (2.18)$$

entailing

$$g_+ + g_- = -g, \quad g_+ - g_- = 2h, \quad (2.19)$$

$$g^+ + g^- = g, \quad g^+ - g^- = 2h, \quad (2.20)$$

$$g_+g_- = -1, \quad g^+g^- = -1, \quad (2.21)$$

and

$$(g_+)^2 - (g_-)^2 = -2gh. \quad (2.22)$$

We have

$$g_+ \xrightleftharpoons{g \rightarrow -g} -g_-, \quad g^+ \xrightleftharpoons{g \rightarrow -g} -g^-, \quad G_+ \xrightleftharpoons{g \rightarrow -g} -G_-, \quad G^+ \xrightleftharpoons{g \rightarrow -g} -G^-. \quad (2.23)$$

Now, by the help of the quadratic forms

$$B(g; R) = - (R^1 - g_-q) (R^1 - g_+q) \equiv q^2 - gqR^1 - (R^1)^2 \quad (2.24)$$

and

$$\hat{B}(g; P) = - (P_1 - g_+\hat{q}) (P_1 - g_-\hat{q}) \equiv \hat{q}^2 - g\hat{q}P_1 - (P_1)^2, \quad (2.25)$$

where  $q = q(R)$  and  $\hat{q} = \hat{q}(P)$ , we introduce the FMF

$$F(g; R) = \sqrt{|B(g; R)|} j(g; R), \quad (2.26)$$

where

$$j(g; R) = \left| \frac{R^1 - g_-q}{R^1 - g_+q} \right|^{-G/4}, \quad (2.27)$$

and the associated Finslerian Hamiltonian function

$$H(g; P) = \sqrt{|\hat{B}(g; P)|} \hat{j}(g; P), \quad (2.28)$$

where

$$\hat{j}(g; P) = \left| \frac{P_1 - g_+\hat{q}}{P_1 - g_-\hat{q}} \right|^{G/4}. \quad (2.29)$$

The functions can also be rewritten as

$$F(g; R) = |R^1 - g_-q|^{G_+/2} |R^1 - g_+q|^{-G_-/2} \quad (2.30)$$

and

$$H(g; P) = |P_1 - g_+\hat{q}|^{G^+/2} |P_1 - g_-\hat{q}|^{-G^-/2}. \quad (2.31)$$

From the FMF we may calculate the associated co-vector

$$R_p \stackrel{\text{def}}{=} F(g; R) \frac{\partial F(g; R)}{\partial R^p} = \frac{1}{2} \frac{\partial F^2(g; R)}{\partial R^p} \quad (2.32)$$

and, then, the Finslerian metric tensor  $\{g_{pq}\}$  according to

$$g_{pq} \stackrel{\text{def}}{=} \frac{\partial R_q}{\partial R^p}. \quad (2.33)$$

By inserting (2.32) in (2.33) one obtains

$$g_{pq}(g; R) = \frac{1}{2} \frac{\partial^2 F^2(g; R)}{\partial R^p \partial R^q}. \quad (2.34)$$

The tensor  $\{h_{pq}\}$  given by the components

$$h_{pq} = g_{pq} - R_p R_q F^{-2}. \quad (2.35)$$

is called *the angular metric tensor*. The entailed identity

$$R^p h_{pq} = 0 \quad (2.36)$$

is useful to apply in many cases. The associated *Cartan torsion tensor*

$$C_{pqr}(g; R) \stackrel{\text{def}}{=} \frac{1}{4} \frac{\partial^3 F^2(g; R)}{\partial R^p \partial R^q \partial R^r} \quad (2.37)$$

can also be obtained as

$$C_{pqr}(g; R) = \frac{1}{2} \frac{\partial g_{pq}(g; R)}{\partial R^r}. \quad (2.38)$$

If we define

$$C_p = C_p^q r = C_{psr} g^{sr}, \quad (2.39)$$

we find that

$$C_p = \frac{\partial \ln \sqrt{|\det(g_{rs})|}}{\partial R^p}. \quad (2.40)$$

The associated Cartan curvature tensor is

$$S_{pqrs} = (C_p^t s C_t^q r - C_p^t r C_t^q s) F^2. \quad (2.41)$$

The Finslerian relations appeared in this way reduce to the pseudo-Euclidean ones in the limit  $g \rightarrow 0$ ; in particular,

$$B|_{g=0} = R_{pq} R^p R^q, \quad \hat{B}|_{g=0} = R^{pq} P_p P_q, \quad (2.42)$$

$$j|_{g=0} = 1, \quad \hat{j}|_{g=0} = 1,$$

$$F|_{g=0} = \sqrt{|R_{pq} R^p R^q|}, \quad H|_{g=0} = \sqrt{|R^{pq} P_p P_q|}, \quad (2.43)$$

$$g_{pq}|_{g=0} = R_{pq}, \quad g^{pq}|_{g=0} = R^{pq}, \quad (2.44)$$

$$C_{pqr}|_{g=0} = 0. \quad (2.45)$$

In the timelike sector,

$$R^0 > 0, \quad g_{-q} < R^1 < g_{+q}, \quad (2.46)$$

we have

$$B > 0 \quad (2.47)$$

and

$$F(g; R) = (R^1 - g_{-}q)^{G_{+}/2} (g_{+}q - R^1)^{-G_{-}/2}, \quad (2.48)$$

and simple direct calculations yield the explicit components

$$R_1 = -(R^1 + gq)\frac{F^2}{B}, \quad R_a = r_{ab}R^b\frac{F^2}{B}, \quad (2.49)$$

$$h_{11} = -q^2\frac{F^2}{B^2}, \quad h_{1a} = r_{ab}R^bR^1\frac{F^2}{B^2}, \quad (2.50)$$

$$h_{ab} = r_{ab}\frac{F^2}{B} - (q - gR^1)r_{ac}R^c r_{bd}R^d\frac{F^2}{qB^2}, \quad (2.51)$$

together with

$$g_{11} = [(R^1 + gq)^2 - q^2]\frac{F^2}{B^2}, \quad g_{1a} = -gqr_{ab}R^b\frac{F^2}{B^2}, \quad (2.52)$$

$$g_{ab} = r_{ab}\frac{F^2}{B} + gr_{ac}R^c r_{bd}R^dR^1\frac{F^2}{qB^2}. \quad (2.53)$$

For the reciprocal components  $\{g^{pq}\}$  we find

$$g^{11}(g; R) = -(gqR^1 + B)F^{-2}, \quad g^{1a}(g; R) = -gqR^aF^{-2}, \quad (2.54)$$

and

$$g^{ab}(g; R) = BF^{-2}r^{ab} - g(R^1 + gq)q^{-1}R^aR^bF^{-2}, \quad (2.55)$$

so that  $g^{pq}g_{qs} = \delta_s^p$ .

It follows that

$$\det(g_{pq}) = -\left(\frac{F^2}{B}\right)^N \det(r_{ab}). \quad (2.56)$$

A convenient way to derive the Finslerian Hamiltonian function (2.31) from the FMF (2.30) is to use the relations

$$\hat{q} = q\frac{F^2}{B}, \quad R^1 = -(R_1 + g\hat{q})\frac{B}{F^2}, \quad (2.57)$$

$$F = H, \quad B\hat{B} = F^4 = F^2H^2, \quad (2.58)$$

and

$$R^1 = -(R_1 + g\hat{q})\frac{H^2}{\hat{B}}, \quad R^a = r^{ab}R_b\frac{H^2}{\hat{B}}. \quad (2.59)$$

Calculating yields

$$F^2 C_p C^p = - \left( \frac{gN}{2} \right)^2 \quad (2.60)$$

$$C_{pqr} = \frac{1}{N} \left[ h_{pq} C_r + h_{pr} C_q + h_{qr} C_p - \frac{1}{C^s C_s} C_p C_q C_r \right]. \quad (2.61)$$

By inserting (2.61) in the tensor (2.41) describing the indicatrix curvature one obtains the result

$$S_{pqrs} = S^* (h_{pr} h_{qs} - h_{pq} h_{rs}) \quad (2.62)$$

with

$$S^* = \frac{1}{4} g^2. \quad (2.63)$$

Since the curvature value  $R_{Ind}$  of the indicatrix (which is defined by the equation  $F = 1$ ) can be found from the formula

$$R_{Ind} = -(1 + S^*) \quad (2.64)$$

(cf. Eq. (2.3.37) in [4]; the negative sign of the signature has been taken into account), we arrive at

**PROPOSITION.** *For the FMF (2.30) the indicatrix is a space of the constant negative curvature*

$$R_{Ind} = - \left( 1 + \frac{1}{4} g^2 \right) \leq -1. \quad (2.65)$$

### 3. ADDITIONAL IMPLICATIONS

With respect to an orthogonal basis, so that  $r_{00} = -r_{22} = \dots = -r_{(N-1)(N-1)} = 1$  (other components of the tensor  $r_{ab}$  vanish), and in terms of the convenient notation

$$v = R^1 / R^0 \equiv v^1, \quad u = \sqrt{(R^2)^2 + \dots + (R^{N-1})^2} / R^0 \equiv v^\perp, \quad (3.1)$$

the timelike sector (2.46) can be specified as

$$R^0 > 0, \quad g_- M < v < g_+ M, \quad (3.2)$$

where

$$M = \sqrt{1 - u^2}. \quad (3.3)$$

We introduce the functions

$$Q(g; v, u) = 1 - v^2 - u^2 - gvM \equiv M^2 - gvM - v^2 > 0 \quad (3.4)$$

and

$$V(g; v, u) = (v - g_- M)^{G_+/2} (g_+ M - v)^{-G_-/2} \quad (3.5)$$

to have

$$B = (R^0)^2 Q \quad (3.6)$$

and

$$F = R^0 V. \quad (3.7)$$

We find

$$\frac{\partial V}{\partial v} = -(gM + v)\frac{V}{Q}, \quad \frac{\partial V}{\partial u} = -u\frac{V}{Q}, \quad (3.8)$$

$$\frac{\partial^2 V}{\partial v^2} = -M^2\frac{V}{Q^2}, \quad \frac{\partial^2 V}{\partial v \partial u} = -vu\frac{V}{Q^2}, \quad \frac{\partial^2 V}{\partial u^2} = -\left(1 - v^2 - \frac{gv}{M}\right)\frac{V}{Q^2}, \quad (3.9)$$

and also,

$$\frac{\partial(V/Q)}{\partial v} = v\frac{V}{Q^2}, \quad \frac{\partial(V/Q)}{\partial u} = u\left(1 - \frac{gv}{M}\right)\frac{V}{Q^2}, \quad (3.10)$$

$$\frac{\partial^2(V/Q)}{\partial v^2} = (1 + 4v^2 - u^2 + gvM)\frac{V}{Q^3}, \quad (3.11)$$

$$\frac{\partial^2(V/Q)}{\partial v \partial u} = \frac{u}{M}(4vM - 2gv^2 + uM - gvu)\frac{V}{Q^3}, \quad (3.12)$$

$$\frac{\partial^2(V/Q)}{\partial u^2} = \left[\left(1 - \frac{gv}{M} + \frac{g vu^2}{M^3}\right)Q + u^2\left(1 - \frac{gv}{M}\right)\left(3 - \frac{2gv}{M}\right)\right]\frac{V}{Q^3}. \quad (3.13)$$

For the function

$$j \stackrel{\text{def}}{=} \frac{V}{\sqrt{Q}} \quad (3.14)$$

we get

$$\frac{\partial j}{\partial v} = -\frac{1}{2}gMj\frac{1}{Q}, \quad \frac{\partial j}{\partial u} = -\frac{1}{2}g\frac{vu}{M}j\frac{1}{Q}. \quad (3.15)$$

The approximation of  $V$  reads

$$V \approx 1 - gv - \frac{1}{2}(v^2 + u^2) - \frac{1}{6}gv^3 - \frac{1}{8}(v^2 + u^2)^2 - \frac{1}{12}g^2v^4 + \dots, \quad (3.16)$$

so that

$$\frac{V}{Q} \approx 1 + \frac{1}{2}(v^2 + u^2) + \frac{1}{6}gv^3 - gvu^2 + \dots. \quad (3.17)$$

#### 4. CO-FRAMEWORK

We similarly proceed for the co-vectors in the time-like sector:

$$\hat{v} = -\frac{P_1}{P_0} = \hat{v}^1, \quad \hat{u} = \sqrt{(P_2)^2 + \dots + (P_{N-1})^2}/P_0 \equiv \hat{v}^\perp, \quad (4.1)$$

$$P_0 > 0, \quad -g_+ \hat{M} < \hat{v} < -g_- \hat{M}, \quad (4.2)$$

$$\hat{Q}(g; \hat{v}, \hat{u}) \stackrel{\text{def}}{=} \hat{B}/(P_0)^2, \quad \hat{M} \stackrel{\text{def}}{=} \hat{q}/P_0. \quad (4.3)$$

We have

$$\hat{M} = \sqrt{1 - \hat{u}^2}, \quad (4.4)$$

$$W \stackrel{\text{def}}{=} H/P_0, \quad (4.5)$$

$$\hat{Q}(g; \hat{v}, \hat{u}) = 1 - \hat{v}^2 - \hat{u}^2 + g\hat{v}\hat{M} \equiv \hat{M}^2 + g\hat{v}\hat{M} - \hat{v}^2 > 0, \quad (4.6)$$

$$\hat{Q}(g; \hat{v}, \hat{u}) = (g_+ \hat{M} + \hat{v})(-g_- \hat{M} - \hat{v}), \quad (4.7)$$

and

$$W(g; \hat{v}, \hat{u}) = \left( g_+ \hat{M} + \hat{v} \right)^{-G_-/2} \left( -g_- \hat{M} - \hat{v} \right)^{G_+/2}. \quad (4.8)$$

By differentiating (4.8) we find

$$\frac{\partial W}{\partial \hat{v}} = (-\hat{v} + g\hat{M}) \frac{W}{\hat{Q}}, \quad \frac{\partial W}{\partial \hat{u}} = -\hat{u} \frac{W}{\hat{Q}}, \quad (4.9)$$

Also,

$$\frac{\partial(W/\hat{Q})}{\partial \hat{v}} = \hat{v} \frac{W}{\hat{Q}^2}, \quad \frac{\partial(W/\hat{Q})}{\partial \hat{u}} = \hat{u} \left( 1 + \frac{g\hat{v}}{\hat{M}} \right) \frac{W}{\hat{Q}^2}. \quad (4.10)$$

For the function

$$\hat{j} \stackrel{\text{def}}{=} \frac{W}{\sqrt{\hat{Q}}} \quad (4.11)$$

we get

$$\frac{\partial \hat{j}}{\partial \hat{v}} = \frac{1}{2} g \hat{M} \hat{j} \frac{1}{\hat{Q}}, \quad \frac{\partial \hat{j}}{\partial \hat{u}} = \frac{1}{2} g \frac{\hat{v} \hat{u}}{\hat{M}} \hat{j} \frac{1}{\hat{Q}}. \quad (4.12)$$

The approximation of  $W$  reads

$$W \approx 1 + g\hat{v} - \frac{1}{2}(\hat{v}^2 + \hat{u}^2) + \frac{1}{6}g\hat{v}^3 - \frac{1}{8}(\hat{v}^2 + \hat{u}^2)^2 - \frac{1}{12}g^2\hat{v}^4 + \dots. \quad (4.13)$$

$$\frac{W}{\hat{Q}} \approx 1 + \frac{1}{2}(\hat{v}^2 + \hat{u}^2) - \frac{1}{6}g\hat{v}^3 + g\hat{v}\hat{u}^2 + \dots \quad (4.14)$$

These differ formally from (3.16)-(3.17) in the negative sign placed at the parameter  $g$ .  
Considering the dispersion relation

$$P_0 W = m, \quad (4.15)$$

we find

$$\frac{\partial P_0}{\partial P_1} = \frac{P_1 + g\sqrt{(P_0)^2 - (P_\perp)^2}}{P_0} \quad (4.16)$$

and

$$\frac{\partial P_0}{\partial P_\perp} = \frac{|P_\perp|}{P_0}. \quad (4.17)$$

So we get the approximation

$$P_0 \approx m + gP_1 + \frac{1}{2m}((P_0)^2 + (P_\perp)^2) + \dots \quad (4.18)$$

## 5. POST-LORENTZIAN TETRADS

The associated Finslerian metric tensor  $g_{pq}(g; R)$  given by the components (2.52) and (2.53) proves to be of space-time signature, that is, be representable as follows:

$$g_{pq}(g; R) = \sum_{P=0}^{N-1} q_{(P)} H_p^{(P)}(g; R) H_q^{(P)}(g; R), \quad q_{(P)} = (1, -1, \dots, -1). \quad (5.1)$$

The contravariant tetrad reciprocal to  $H_p^{(P)}(g; R)$  will be denoted by  $H_{(P)}^p(g; R)$ , so that  $H_p^{(P)}(g; R) H_{(Q)}^p(g; R) = \delta_Q^P$ . The Finslerian tetrad  $H_p^{(P)}(g; R)$  is a geometrical representation for a RS of a RF. If

$$H_{(0)}^p(g; R) || R^p, \quad (5.2)$$

we call the tetrad proper kinematical, for in such a case the vector  $H_{(0)}^p$  indicates the four-dimensional direction along which the local observer is moving.

On using (2.52)-(2.55) and (3.1)-(3.6), the attentive calculation of tetrads obeying (5.1)-(5.2) results in the following list:

$$H_{(0)}^0(g; v, u) = \frac{1}{V(g; v, u)}, \quad H_{(0)}^1(g; v, u) = \frac{v}{V(g; v, u)}, \quad H_{(0)}^2(g; v, u) = \frac{u}{V(g; v, u)}, \quad (5.3)$$

$$H_{(1)}^0(g; v, u) = \left(\frac{v}{M} + g\right) \frac{1}{V(g; v, u)}, \quad H_{(1)}^1(g; v, u) = \frac{M}{V(g; v, u)}, \quad H_{(1)}^2(g; v, u) = \left(\frac{v}{M} + g\right) \frac{u}{V(g; v, u)}, \quad (5.4)$$

$$H_{(2)}^0(g; v, u) = \frac{u}{M} \frac{\sqrt{Q(g; v, u)}}{V(g; v, u)}, \quad H_{(2)}^1(g; v, u) = 0, \quad H_{(2)}^2(g; v, u) = \frac{1}{M} \frac{\sqrt{Q(g; v, u)}}{V(g; v, u)}, \quad (5.5)$$

$$H_{(3)}^0(g; v, u) = H_{(3)}^1(g; v, u) = H_{(3)}^2(g; v, u) = 0, \quad H_{(3)}^3(g; v, u) = \frac{\sqrt{Q(g; v, u)}}{V(g; v, u)}, \quad (5.6)$$

$$H_{(0)}^3(g; v, u) = H_{(1)}^3(g; v, u) = H_{(2)}^3(g; v, u) = 0, \quad (5.7)$$

together with

$$H_0^{(0)}(g; v, u) = \frac{V(g; v, u)}{Q(g; v, u)}, \quad H_1^{(0)}(g; v, u) = -(v+gM)\frac{V(g; v, u)}{Q(g; v, u)}, \quad H_2^{(0)}(g; v, u) = -u\frac{V(g; v, u)}{Q(g; v, u)}, \quad (5.8)$$

$$H_0^{(1)}(g; v, u) = -\frac{v}{M}\frac{V(g; v, u)}{Q(g; v, u)}, \quad H_1^{(1)}(g; v, u) = M\frac{V(g; v, u)}{Q(g; v, u)}, \quad H_2^{(1)}(g; v, u) = \frac{vu}{M}\frac{V(g; v, u)}{Q(g; v, u)}, \quad (5.9)$$

$$H_0^{(2)}(g; v, u) = -\frac{u}{M}\frac{V(g; v, u)}{\sqrt{Q(g; v, u)}}, \quad H_1^{(2)}(g; v, u) = 0, \quad H_2^{(2)}(g; v, u) = \frac{1}{M}\frac{V(g; v, u)}{\sqrt{Q(g; v, u)}}, \quad (5.10)$$

$$H_0^{(3)}(g; v, u) = H_1^{(3)}(g; v, u) = H_2^{(3)}(g; v, u) = 0, \quad H_3^{(3)}(g; v, u) = \frac{V(g; v, u)}{\sqrt{Q(g; v, u)}}, \quad (5.11)$$

$$H_3^{(0)}(g; v, u) = H_3^{(1)}(g; v, u) = H_3^{(2)}(g; v, u) = 0. \quad (5.12)$$

## 6. EXPLICATED FINSLERIAN KINEMATIC TRANSFORMATIONS

If  $A^p$  denotes a four-dimensional vector that represents the motion velocity of the RF  $S_{\{v, u\}}$  with respect to the RF  $\Sigma$ , so that

$$v \equiv v^1 = \frac{\Delta A^1}{\Delta A^0}, \quad u \equiv v^2 = \frac{\Delta A^2}{\Delta A^0}, \quad v^3 = 0 \quad (6.1)$$

(when motion is going in the plane  $R^1 \times R^2$ ) and the time-like sector is chosen, in accordance with (3.2) and (3.4), we may use the tetrads of preceding section to connect the RFs:

$$R^p = H_{(Q)}^p(g; A)r^{(Q)} \quad (6.2)$$

(the tetrads play the role of the reference systems associated with the RFs). In this way, straightforward calculations lead to the sought generalized kinematic transformations.

Namely, on re-labeling  $r^{(Q)} = (t, x, y, z)$  and using the list (5.3)-(5.12), with  $A$  substituted with  $R$ , we get explicitly

$$R^0 = \frac{1}{V(g; v, u)} \left[ t + \left( \frac{v}{M} + g \right) x + \frac{u}{M} \sqrt{Q(g; v, u)} y \right], \quad (6.3)$$

$$R^1 = \frac{1}{V(g; v, u)}(vt + Mx), \quad (6.4)$$

$$R^2 = \frac{1}{V(g; v, u)} \left[ ut + \left( \frac{v}{M} + g \right) ux + \frac{1}{M} \sqrt{Q(g; v, u)} y \right], \quad (6.5)$$

$$R^3 = \frac{\sqrt{Q(g; v, u)}}{V(g; v, u)} z, \quad (6.6)$$

which inverse reads

$$t = \frac{V(g; v, u)}{Q(g; v, u)} \left[ R^0 - (v + gM)R^1 - uR^2 \right], \quad (6.7)$$

$$x = \frac{V(g; v, u)}{Q(g; v, u)} \left[ -\frac{v}{M} R^0 + MR^1 + \frac{vu}{M} R^2 \right], \quad (6.8)$$

$$y = \frac{V(g; v, u)}{\sqrt{Q(g; v, u)}} \frac{1}{M} (-uR^0 + R^2), \quad (6.9)$$

$$z = \frac{V(g; v, u)}{\sqrt{Q(g; v, u)}} R^3. \quad (6.10)$$

Here,  $Q$  and  $V$  are the functions (3.4) and (3.5), respectively;

$$\{R^0, R^1, R^2, R^3\} \in \Sigma, \quad \{t, x, y, z\} \in S_{\{v, u\}}, \quad (6.11)$$

where  $\Sigma$  is the input preferred rest frame and  $S_{\{v, u\}}$  is an inertial RF moving with the velocity  $\{v^1 = v, v^2 = u, v^3 = 0\}$  relative to  $\Sigma$ ; the instantaneously common origin of the frames being implied.

Similarly for the momenta,

$$P_0 = \frac{V(g; v, u)}{Q(g; v, u)} \left[ p_0 - \frac{v}{M} p_1 - \frac{u}{M} \sqrt{Q(g; v, u)} p_2 \right], \quad (6.12)$$

$$P_1 = \frac{V(g; v, u)}{Q(g; v, u)} \left[ -(v + gM)p_0 + Mp_1 \right], \quad (6.13)$$

$$P_2 = \frac{V(g; v, u)}{Q(g; v, u)} \left[ -up_0 + \frac{vu}{M} p_1 + \frac{1}{M} \sqrt{Q(g; v, u)} p_2 \right], \quad (6.14)$$

$$P_3 = \frac{V(g; v, u)}{\sqrt{Q(g; v, u)}} p_3, \quad (6.15)$$

and its inverse

$$p_0 = \frac{1}{V(g; v, u)} (P_0 + vP_1 + uP_2), \quad (6.16)$$

$$p_1 = \frac{1}{V(g; v, u)} \left[ \left( \frac{v}{M} + g \right) P_0 + M P_1 + \left( \frac{v}{M} + g \right) u P_2 \right], \quad (6.17)$$

$$p_2 = \frac{\sqrt{Q(g; v, u)}}{V(g; v, u)} \frac{1}{M} (u P_0 + P_2) \quad (6.18)$$

$$p_3 = \frac{\sqrt{Q(g; v, u)}}{V(g; v, u)} P_3, \quad (6.19)$$

where

$$\{P_0, P_1, P_2, P_3\} \in \Sigma, \quad \{p_0, p_1, p_2, p_3\} \in S_{\{v, u\}}. \quad (6.20)$$

The invariance of the contraction:

$$R^0 P_0 + R^1 P_1 + R^2 P_2 + R^3 P_3 = t p_0 + x p_1 + y p_2 + z p_3 \quad (6.21)$$

can be verified directly.

Inversely, one may postulate (6.21) to explicate (6.12)-(6.19) from (6.3)-(6.10).

The transformations (6.3)-(6.10) can also be rewritten in the form

$$t = a(g; v, u) R^0 + \epsilon(g; v, u) x + \epsilon_2(g; v, u) y, \quad (6.22)$$

$$x = b(g; v, u) R^1 + b_1(g; v, u) R^0 + b_2(g; v, u) R^2, \quad (6.23)$$

$$y = \frac{d(g; v, u)}{\sqrt{1 - u^2}} (R^2 - u R^0), \quad (6.24)$$

and

$$z = d(g; v, u) R^3, \quad (6.25)$$

with the kinematic coefficients

$$a = V(g; v, u), \quad (6.26)$$

$$\epsilon = -\frac{v}{\sqrt{1 - u^2}} + g, \quad (6.27)$$

$$\epsilon_2 = -\frac{u}{\sqrt{1 - u^2}} \sqrt{Q(g; v, u)}, \quad (6.28)$$

$$b = \frac{V(g; v, u)}{Q(g; v, u)} \sqrt{1 - u^2}, \quad b_1 = -\frac{V(g; v, u)}{Q(g; v, u)} \frac{v}{\sqrt{1 - u^2}}, \quad (6.29)$$

$$b_2 = \frac{V(g; v, u)}{Q(g; v, u)} \frac{vu}{\sqrt{1 - u^2}}, \quad (6.30)$$

and

$$d = \frac{V(g; v, u)}{\sqrt{Q(g; v, u)}}. \quad (6.31)$$

Whenever  $|v| \ll 1$  and  $|u| \ll 1$ , the low-velocity approximations

$$a(g; v, u) \approx 1 - gv - \frac{1}{2}(v^2 + u^2) - \frac{1}{6}gv^3 - \frac{1}{8}(v^2 + u^2)^2 - \frac{1}{12}g^2v^4 + ..., \quad (6.32)$$

$$b(g; v, u) \approx 1 + \frac{1}{2}v^2 + \frac{1}{6}gv^3 - gvu^2 + ..., \quad (6.33)$$

$$d(g; v, u) \approx 1 - \frac{1}{2}gv + ..., \quad (6.34)$$

$$\epsilon(g; v, u) \approx g - v - \frac{1}{2}vu^2 + ..., \quad (6.35)$$

$$\epsilon_2(g; v, u) \approx -u + \frac{1}{2}g vu + ..., \quad (6.36)$$

$$b_1(g; v, u) \approx -v(1 + \frac{1}{2}v^2 + u^2 + \frac{1}{6}gv^3 - gvu^2 + ...), \quad (6.37)$$

$$b_2(g; v, u) \approx vu(1 + \frac{1}{2}v^2 + u^2 + \frac{1}{6}gv^3 - gvu^2 + ...) \quad (6.38)$$

are obtained.

The Finslerian parameter  $g$  is characteristic, so that in the limit  $g \rightarrow 0$  the above kinematic transformations (6.3)-(6.10) reduce exactly to the ordinary special-relativistic Lorentz transformations

$$R^0 = \frac{1}{\sqrt{1 - v^2 - u^2}}t + \frac{v}{\sqrt{1 - v^2 - u^2}}x, \quad R^1 = \frac{v}{\sqrt{1 - v^2 - u^2}}t + \frac{1}{\sqrt{1 - v^2 - u^2}}x, \quad (6.39)$$

$$R^2 = y, \quad R^3 = z, \quad (6.40)$$

$$t = \frac{1}{\sqrt{1 - v^2 - u^2}}R^0 - \frac{v}{\sqrt{1 - v^2 - u^2}}R^1, \quad x = -\frac{v}{\sqrt{1 - v^2 - u^2}}R^0 + \frac{1}{\sqrt{1 - v^2 - u^2}}R^1, \quad (6.41)$$

and

$$P_0 = \frac{1}{\sqrt{1-v^2-u^2}} p_0 - \frac{v}{\sqrt{1-v^2-u^2}} p_1, \quad P_1 = -\frac{v}{\sqrt{1-v^2-u^2}} p_0 + \frac{1}{\sqrt{1-v^2-u^2}} p_1, \quad (6.42)$$

$$P_2 = p_2, \quad P_3 = p_3, \quad (6.43)$$

$$p_0 = \frac{1}{\sqrt{1-v^2-u^2}} P_0 + \frac{v}{\sqrt{1-v^2-u^2}} P_1, \quad p_1 = \frac{v}{\sqrt{1-v^2-u^2}} P_0 + \frac{1}{\sqrt{1-v^2-u^2}} P_1. \quad (6.44)$$

## 7. Concluding Remarks

We have served the purpose of organizing the FG-theoretical framework which goes beyond square-root metric and admits a single preferred spatial direction, thereby offering rather general theoretical and methodological basis for anisotropic relativistic applications.

Apart from relevant generalized-kinematic applications, which can be developed with the aid of the post-Lorentzian Finslerian kinematic transformations (6.3)-(6.10), there are also opening up extended general-relativistic prospects, as being suggested by the very form of the FMF (2.30) under study. Indeed, the later function invites lifting the Schwarzschild line element

$$\left( ds^{Schwarzschildian}_{isotropic} \right)^2 = (1-2U)c^2(dt)^2 - \frac{1}{1-2U}(dr)^2 - r^2((d\theta)^2 + \sin^2\theta(d\phi)^2) \quad (7.1)$$

( $U = KM_0/c^2r$  and  $K$  is the universal gravitational constant; the spherical coordinates have been used; the gravitational field is created by a mass  $M_0$  at the origin) to the advanced-Finslerian level by introducing the extension

$$\left( ds^{Finslerian}_{anisotropic} \right)^2 = \left[ \frac{1}{\sqrt{1-2U}} dr - g_- \sqrt{(1-2U)c^2(dt)^2 - r^2((d\theta)^2 + \sin^2\theta(d\phi)^2)} \right]^{G_+} \times \quad (7.2)$$

$$\left[ g_+ \sqrt{(1-2U)c^2(dt)^2 - r^2((d\theta)^2 + \sin^2\theta(d\phi)^2)} - \frac{1}{\sqrt{1-2U}} dr \right]^{-G_-} \quad (7.3)$$

in accord with the correspondence principle to hold true in the limit  $g \rightarrow 0$ :

$$ds^{Finslerian}_{anisotropic}|_{g=0} = ds^{Schwarzschildian}_{isotropic} \quad (7.4)$$

Alternatively, for the FG-isotropic way developed in the previous papers [5-9], one gets

$$\left( ds^{Finslerian}_{isotropic} \right)^2 = \left[ \sqrt{1-2U}cdt + g_- \sqrt{\frac{1}{1-2U}(dr)^2 + r^2((d\theta)^2 + \sin^2\theta(d\phi)^2)} \right]^{G_+} \times \quad (7.5)$$

$$\left[ \sqrt{1 - 2U} c dt + g_+ \sqrt{\frac{1}{1 - 2U} (dr)^2 + r^2 ((d\theta)^2 + \sin^2 \theta (d\phi)^2)} \right]^{-G_-}, \quad (7.6)$$

for which again

$$ds_{isotropic}^{Finslerian} |_{g=0} = ds_{isotropic}^{Schwarzschildian}. \quad (7.7)$$

All the Finslerian relativistic relations derived can, in principle, be verified experimentally to put empirical limitations on the characteristic Finslerian parameter  $g$ . The corresponding applications will be made in a special work to be published elsewhere.

Generally, as the flow of publications concerning post-Lorentzian effects is increasing steadily, and the reasons for such effects are claimed starting from various interesting and important standpoints [10-32], it might be useful to coordinate relevant attempts with the possibility of constructing post-Lorentzian extensions in a FG-consistent geometrical rigorous way.

## References

- [1] H. Minkowski: *Raum und Zeit — Phys. Z.* **10** (1909), 104.
- [2] H. Minkowski: *Das Relativitätsprinzip — Ann. Physik.* **47** (1915), 927.
- [3] H. Rund: *The Differential Geometry of Finsler Spaces*, Springer, Berlin 1959.
- [4] G.S. Asanov: *Finsler Geometry, Relativity and Gauge Theories*, D. Reidel Publ. Comp., Dordrecht 1985.
- [5] G.S. Asanov: *Rep. Math. Phys.* **39** (1997), 69; **41** (1998), 117; **46** (2000), 383; **47** (2001), 323.
- [6] G.S. Asanov: *Moscow University Physics Bulletin* **49**(1) (1994), 18; **49**(2) (1994), 11; **51**(1) (1996), 15; **51**(2) (1996) 6; **51**(3) (1996) 1; **53**(1) (1998), 15.
- [7] G.S. Asanov: The Finsler-type recasting of Lorentz transformations. In: Proceedings of Conference “Physical Interpretation of Relativity Theory”, September 15-20, London, Sunderland, 2000, pp. 1-24.
- [8] G.S. Asanov: Can neutrinos and high-energy particles test Finsler metric of space-time? arXiv:hep-ph/0009305, 27 Sep. 2000.
- [9] G.S. Asanov: *Found. Phys. Lett.* **15** (2002), 199.
- [10] Proceedings of Conference “Physical Interpretation of Relativity Theory”, September 15-20, London, Sunderland, 2000.
- [11] J.A. Winnie: *Philos. Sci.* **37** (1970), 81, 223.
- [12] G. Spavieri: *Found. Phys. Lett.* **1** (1988), 373.
- [13] S.J. Prokhovnik and W.T. Morris: *Found. Phys.* **19** (1989), 531.
- [14] R. Mansouri and R. Sexl: *Gen. Rel. Grav.* **8** (1977), 496, 515, 809.
- [15] S.J. Prokhovnik: *Found. Phys.* **3** (1973), 351; **9** (1979), 883; **10** (1980, 197; **19** (1989), 541; *J. Australian Math. Soc.* **5**(2) (1965), 273; **6**(1) (1966), 101.
- [16] S.J. Prokhovnik: *The Logic of Special Relativity*, Cambridge University Press, Cambridge, 1967.
- [17] A.K.A. Maciel and J. Tiomno: *Found. Phys.* **19** (1989), 505 and 521.
- [18] G. Spavieri: *Phys. Rev.* **A34** (1986), 1708.
- [19] W.A. Rodrigues and J. Tiomno: *Found. Phys.* **15** (1985), 945.
- [20] Y.Z. Zhang: *Gen. Rel. Grav.* **27** (1995), 475.
- [21] F. Selleri: *Chin. J. Eng. Elect.* **6** (1995), 25; *Found. Phys. Lett.* **9**(1) (1997), 73; *Found. Phys.* **26** (1996), 641.
- [22] F. Selleri: “Space and Time should be preferred to Spacetime, 1 and 2”. In: Redshift and gravitation in a relativistic universe (K. Rudnicki, ed.), Apeiron, Montreal (2001), pp. 63-71 and 81-94.
- [23] M.D. Gabriel and M.P. Haugan: *Phys. Rev.* **D41** (1990), 2943.
- [24] M.P. Haugan and C.M. Will: *Physics Today* **40** (1987), 69.
- [25] C.M. Will: *Phys. Rev.* **D45** (1992), 403.
- [26] S. Marinov: *Czech. J. Phys.* **24** (1974), 965; *Found. Phys.* **9** (1979), 445; *Gen. Rel. Grav.* **12** (1980), 57.
- [27] E.M. Kelly: *Found. Phys.* **14**, (1984), 705; **15** (1985), 333.
- [28] A. Brillet and J.L. Hall: *Phys. Rev. Lett.* **42** (1979), 549.
- [29] S. Coleman and S.L. Glashow: Cosmic ray and neutrino tests of special relativity *Harvard University Report No. HUTP-97/A008*, hep-ph/9703240, 1-5.
- [30] S.L. Glashow, A. Halprin, P.L. Krastev, C.N. Leung, and J. Pantaleone: Remarks on neutrino tests of special relativity. *Phys. Rev.D* **56**, N 4 (1997), 2433-2434.
- [31] R. Anderson, I. Vetharaniam and G.E. Stedman: *Phys. Reports* **295**, (1998), 93-180.
- [32] E. Minguzzi: *Found. Phys. Lett.* **15** (2002), 153.